

Asymptotic density of Motzkin numbers modulo small primes

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Abstract

We establish the asymptotic density of the Motzkin numbers modulo 2, 4, 8, 3 and 5.

1 Introduction

The Motzkin numbers M_n are defined by

$$M_n := \sum_{k \geq 0} \binom{n}{2k} C_k$$

where C_k are the Catalan numbers.

There has been some work in recent years on analysing the Motzkin numbers M_n modulo primes and prime powers. This work has often been done in concert with and using the same methods as work analysing the Catalan numbers. Deutsch and Sagan [1] provided a characterisation of Motzkin numbers divisible by 2, 4 and 5. They also provided a complete characterisation of the Motzkin numbers modulo 3 and showed that no Motzkin number is divisible by 8. Eu, Liu and Yeh [2] reproved some of these results and extended them to include criteria for when M_n is congruent to $\{2, 4, 6\} \pmod{8}$. Krattenthaler and Müller [4] established identities for the Motzkin numbers modulo higher powers of 3 which include the modulo 3 result of [1] as a special case. Krattenthaler and Müller [3] have more recently extended this work to a full characterisation of $M_n \pmod{8}$ in terms of the binary expansion of n . Their characterisation is rather elaborate and less susceptible to analysis than that provided in [2]. The results in [4] and [3] are obtained by expressing the generating function of M_n as a polynomial involving a special function. Rowland and Yassawi [5] investigated M_n in the general setting of automatic sequences. The values of M_n (as well as other sequences) modulo prime powers can be computed via automata.

Rowland and Yassawi provided algorithms for creating the relevant automata. They established results for M_n modulo small prime powers, including a full characterisation of M_n modulo 8 (modulo 5^2 and 13^2 are available from Rowlands website). They also established that 0 is a forbidden residue for M_n modulo 8, 5^2 and 13^2 . In theory the automata can be constructed for any prime power but computing power and memory quickly becomes a barrier. For example, the automata for M_n modulo 13^2 has over 2000 states. Rowland and Yassawi also went on to describe a method for obtaining asymptotic densities of M_n .

We will use the above results to establish asymptotic densities of M_n modulo 2, 4, 8, 3 and 5. Here, the asymptotic density of a subset S of \mathbb{N} is defined to be

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{n \in S : n \leq N\}$$

if the limit exists, where $\#S$ is the number of elements in a set S . In contrast to the results for the Catalan numbers C_n , the set of Motzkin numbers congruent to 0 mod n is not expected to have asymptotic density 1 for a general $n \in \mathbb{N}$. The results here show that this expectation holds for small values of n .

2 Asymptotic density of certain forms of numbers

The main method in the literature of characterising $M_n \pmod q$ is to divide the natural numbers into classes of the form

$$S(q, r, s, t) = \{(qi + r)q^{sj+t} + c : i, j \in \mathbb{N}\}$$

for various choices of r, s, t and c . It will therefore be useful to know how these types of sets behave asymptotically. We can disregard the c term as this does not change the asymptotic behaviour. So the set of interest is

$$S(q, r, s, t) = \{(qi + r)q^{sj+t} : i, j \in \mathbb{N}\} \quad (1)$$

for integers q, r, s, t .

Theorem 1. *Let $q, r, s, t \in \mathbb{Z}$ with $q, s > 0$, $t \geq 0$ and $0 \leq r < q$. Then the asymptotic density of the set S is $(q^{t+1-s}(q^s - 1))^{-1}$.*

Proof. We have, for fixed $j \geq 0$,

$$\#\{i \geq 0 : (qi + r)q^{sj+t} \leq N\} = \frac{N}{q^{sj+t+1}} - \frac{r}{q} - E(j, N, q, r, s, t)$$

where $0 \leq E(j, N, q, r, s, t) < 1$ is an error term introduced by not rounding down to the nearest integer. So, letting

$$U(N, s, t) := \left\lfloor \frac{\log_q(N) - t - 1}{s} \right\rfloor$$

, we have

$$\begin{aligned}
& \#\{n < N : n = (qi + r) q^{sj+t} \text{ for some } i, j \in \mathbb{N}\} \\
&= \sum_{j \geq 0} \left(\frac{N}{q^{sj+t+1}} - \frac{r}{q} - E(j, N, q, r, s, t) \right) \\
&= \sum_{j=0}^U \left(\frac{N}{q^{sj+t+1}} - \frac{r}{q} - E(j, N, q, r, s, t) \right) \\
&= \frac{N}{q^{t+1}} \sum_{j=0}^U \left(\frac{1}{q^s} \right)^j - E'(N, q, r, s, t)
\end{aligned}$$

where the new error term $E'(N, q, r, s, t)$ satisfies

$$0 < E'(N, q, r, s, t) < 2(U + 1).$$

Then

$$\begin{aligned}
& \#\{n < N : n = (qi + r) q^{sj+t} \text{ for some } i, j \in \mathbb{N}\} \\
&= \left(\frac{N}{q^{t+1}} \right) \left(1 - \left(\frac{1}{q^s} \right)^{U+1} \right) \left(1 - \frac{1}{q^s} \right)^{-1} - E'.
\end{aligned}$$

Since $\lim_{N \rightarrow \infty} \frac{E'(N, q, r, s, t)}{N} = 0$ and $\lim_{N \rightarrow \infty} \frac{1}{N} \left(\frac{1}{q^s} \right)^{U+1} = 0$ we have

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \#\{n < N : n = (qi + r) q^{sj+t} \text{ for some } i, j \in \mathbb{N}\} \\
&= \left(q^{t+1-s} (q^s - 1) \right)^{-1}.
\end{aligned}$$

□

Remark. Sometimes we will need to consider the set

$$S'(q, r, s, t) = \{(qi + r) q^{sj+t} : i, j \in \mathbb{N}, j \geq 1\} \quad (2)$$

for integers q, r, s, t . The asymptotic density of the set S' can be derived from theorem 1 as $(q^{t+1}(q^s - 1))^{-1}$.

3 Motzkin numbers modulo 2, 4 and 8

The following result is established in [2]

Theorem 2. (Theorem 5.5 of [2]). The n th Motzkin number M_n is even if and only if

$$n = (4i + \epsilon)4^{j+1} - \delta \text{ for } i, j \in \mathbb{N}, \epsilon \in \{1, 3\} \text{ and } \delta \in \{1, 2\}.$$

Moreover, we have

$$M_n \equiv 4 \pmod{8} \text{ if } (\epsilon, \delta) = (1, 1) \text{ or } (3, 2)$$

$$M_n \equiv 4y + 2 \pmod{8} \text{ if } (\epsilon, \delta) = (1, 2) \text{ or } (3, 1)$$

where y is the number of digit 1s in the base 2 representation of $4i + \epsilon - 1$.

Remark. The 4 choices of (ϵ, δ) in the above theorem give 4 disjoint sets of numbers $n = (4i + \epsilon)4^{j+1} - \delta$.

Theorem 3. Each of the 4 disjoint sets defined by the choice of (ϵ, δ) in Theorem 2 has asymptotic density $\frac{1}{12}$ in the natural numbers.

Proof. Use the result of Theorem 1 for the set S with $q = 4, r = \epsilon, s = 1, t = 1$. \square

Corollary 4. The asymptotic density of

$$\{n < N : M_n \equiv 0 \pmod{2}\} \text{ is } \frac{1}{3}.$$

The asymptotic density of

$$\{n < N : M_n \equiv 4 \pmod{8}\} \text{ is } \frac{1}{6}.$$

The asymptotic density of each the sets

$$\{n < N : M_n \equiv 2 \pmod{8}\} \text{ and } \{n < N : M_n \equiv 6 \pmod{8}\} \text{ is } \frac{1}{12}.$$

Proof. The first 2 statements of the corollary follow immediately from theorem 2 and theorem 3. The third statement follows from the observation that the numbers of 1's in the base 2 expansion of i is equally likely to be odd or even and therefore the same applies to the the number of 1's in the base 2 expansion of $4i + \epsilon - 1$. Since the asymptotic density of the 2 sets combined is $\frac{1}{6}$ (from theorem 3), each of the two sets has asymptotic density $\frac{1}{12}$. \square

Remark. Rowland and Yassawi [5] proved the first two results of the corollary and also established that the asymptotic density of the sets of M_n congruent to 2 modulo 4 is $\frac{1}{6}$.

4 Motzkin numbers modulo 5

The following result is established in [1]

Theorem 5. (Theorem 5.4 of [1]). The Motzkin number M_n is divisible by 5 if and only if n is one of the following forms

$$(5i+1)5^{2j}-2, (5i+2)5^{2j-1}-1, (5i+3)5^{2j-1}-2, (5i+4)5^{2j}-1$$

where $i, j \in \mathbb{N}$ and $j \geq 1$.

Theorem 6. The asymptotic density of numbers of the first form in theorem 5 is $\frac{1}{120}$. Numbers of the fourth form also have asymptotic density $\frac{1}{120}$. The asymptotic density of numbers of the second and third forms in theorem 5 is $\frac{1}{24}$ each.

Proof. Firstly consider numbers of the form $(5i+r)5^{2j}-2$. As we are interested in asymptotic density it is enough to look at numbers of the form $(5i+r)5^{2j}$. We can now use the remark 2 at the end of theorem 1 for the set S' with $q=5, s=2$ and $t=0$. From the remark the asymptotic density of the set

$$\{n \in \mathbb{N} : n = (5i+r)5^{2j} \text{ with } i, j \in \mathbb{N} \text{ and } j \geq 1\}$$

is $(5 \times (5^2 - 1))^{-1} = \frac{1}{120}$. For numbers of the second and third forms we shift the j index so that it starts from 0 and use theorem 1 for the set S with $q=5, s=2$ and $t=1$. From theorem 1 the asymptotic density of the set

$$\{n \in \mathbb{N} : n = (5i+r)5^{2j+1} \text{ with } i, j \in \mathbb{N} \text{ and } j \geq 0\}$$

is $(5^0(5^2 - 1))^{-1} = \frac{1}{24}$. □

Corollary 7. The asymptotic density of $\#\{n < N : M_n \equiv 0 \pmod{5}\}$ is $\frac{1}{10}$.

Proof. The corollary follows immediately from theorem 5 and theorem 6 and the disjointness of the 4 forms of integers listed in theorem 5. □

Remark. Numerical tests also show that roughly 22.5% of Motzkin numbers are congruent to each of $1, 2, 3, 4 \pmod{5}$.

5 Motzkin numbers modulo 3

The structure of the Motzkin numbers modulo 3 is based on a set $T(01)$ which was defined by Deutsh and Sagan in [1]. The set $T(01)$ is the set of natural numbers which have a base 3 expansion containing only the digits 0 and 1. The following theorem from [1] will be used in this section.

Theorem 8. (Corollary 4.10 of [1]). The Motzkin numbers satisfy

$$\begin{aligned} M_n &\equiv -1 \pmod{3} & \text{if } n \in 3T(01) - 1, \\ M_n &\equiv 1 \pmod{3} & \text{if } n \in 3T(01) \text{ or } n \in 3T(01) - 2, \\ M_n &\equiv 0 \pmod{3} & \text{otherwise.} \end{aligned}$$

We will first examine the nature of the set $T(01)$. We have,

Theorem 9. The asymptotic density of the set $T(01)$ is zero.

Proof. Let $N \in \mathbb{N}$ and choose $k \in \mathbb{N} : 3^k \leq N < 3^{k+1}$. Then $k = \lfloor \log_3(N) \rfloor$ and

$$\begin{aligned} \frac{1}{N} \# \{n \leq N : n \in T(01)\} &\leq \frac{2^{k+1}}{N} \\ &\leq \frac{2^{k+1}}{3^k} \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

□

Theorem 10. The asymptotic density of the set $\{n \leq N : M_n \equiv 0 \pmod{3}\}$ is 1.

Proof. Since the asymptotic density of $T(01)$ is zero, so is the asymptotic density of the sets $3T(01) - k$ for $k \in \{0, 1, 2\}$. Therefore theorem 8 implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{n \leq N : M_n \equiv \pm 1 \pmod{3}\} = 0$$

and the result follows. □

References

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